Sydney Boys' High School

MOORE PARK, SURRY HILLS

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 1998

MATHEMATICS

4 UNIT

Time allowed: 3 Hours (plus five minutes reading time)

Total Marks: 120

Examiner: C.Kourtesis

DIRECTIONS TO CANDIDATES

ALL questions may be attempted.

All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.

Standard integrals are provided. Approved calculators may be used.

Each question attempted is to be returned on a separate answer sheet. Each answer sheet must show your name.

Additional answer sheets may be obtained from the supervisor upon request.

NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate Examination Paper for this subject.

Question 1. (15 marks) (Start a new answer sheet.)

(a) Find

(i) $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}}$ **2**

(ii) $\displaystyle\int \frac{1}{x}\left(1+\ln x\right)^5 dx$ by using the substitution $u = 1+\ln x$. **3**

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$. **3**

(c) Evaluate $\displaystyle\int_0^5 \frac{x}{\sqrt{x+4}}dx$. **3**

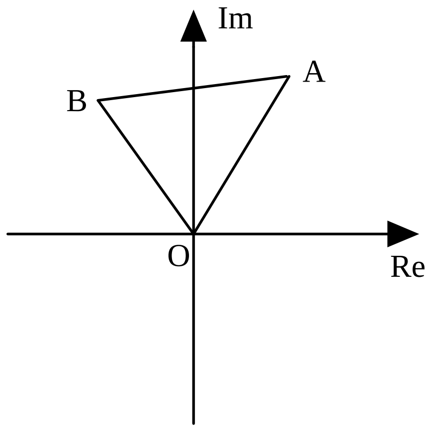
(d) Evaluate $\displaystyle\int_0^2 \frac{8}{(x+2)\left(x^2+4\right)}dx$. **4**

Question 2. (15 marks) (Start a new answer sheet.)

(a) (i) Express $z = 2 + 2i$ in modulus-argument form. **2**

(ii) Hence write z^8 in the form $a + ib$ where a and b are real. **2**

(b) **2**



In the Argand diagram point A corresponds to the complex number $1 + i\sqrt{2}$. If the origin, A, and B are the vertices of an equilateral triangle what complex number corresponds to the vertex B?

(c) Find the locus of z if $\text{Re}(z) = |z|$. **2**

(d) If a, b, c, d are real, and $ad > bc$, show that $\text{Im}\left(\frac{a+ib}{c+id}\right) < 0$. **2**

(e) If P represents the complex number z, where z satisfies

$$|z-2| = 2 \text{ and } 0 < \arg z < \frac{\pi}{2}:$$

(i) Show that $|z^2 - 2z| = 2|z|$. **2**

(ii) Find the value of k (a real number) if $\arg(z-2) = k\arg(z^2 - 2z)$. **3**

Question 3. (15 marks) (Start a new answer sheet.)

(a) The equation $2x^3+5x+1=0$ has roots α,β,γ. Evaluate $\alpha^3+\beta^3+\gamma^3$. **2**

(b) Given the polynomial $P(x)=2x^3-4x^2+mx+n$ where m and n are real numbers:

(i) Find the values of m and n if $1+i$ is a root of $P(x)=0$. **3**

(ii) Find the zeros of $P(x)$. **1**

(c) A monic cubic polynomial when divided by x^2-9 leaves a remainder of $x+8$ and when divided by x leaves a remainder of -4. Express the polynomial in the form **3**

$$ax^3+bx^2+cx+d.$$

(d) (i) By letting $c=\cos\theta$, show that the equation $\cos 4\theta=\cos 3\theta$ can be expressed in the form $8c^4-4c^3-8c^2+3c+1=0$. **2**

(ii) Show that $\theta=\dfrac{2n\pi}{7}$, where n is an integer, satisfies the equation $\cos 4\theta=\cos 3\theta$. **2**

(iii) Using parts (i) and (ii) above, find the equation whose roots are **2**

$\cos\dfrac{2\pi}{7}$, $\cos\dfrac{4\pi}{7}$, $\cos\dfrac{6\pi}{7}$, expressing your answer in polynomial form.

Question 4. (15 marks) (Start a new answer sheet.)

(a) If $f(x)=\dfrac{2-x}{2+x}$ sketch the graphs of:

(i) $y=f(x)$ **2**

(ii) $y=\left[f(x)\right]^2$ by finding the turning points. **3**

(iii) $y=\sqrt{f(x)}$ **2**

(iv) $y=\ln\left[f(x)\right]$ **2**

(b) (i) On the same set of axes shade in the region satisfying both **2**

$$x^2+y^2\leq 1 \text{ and } x^2\leq \frac{8}{3}y.$$

(ii) The area in part (i) is rotated about the y axis through one complete revolution. Using the cylindrical shell method find the volume of the solid generated. **4**

Question 5. (15 marks) (Start a new answer sheet.)

(a) An ellipse has equation $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$.

(i) Find the eccentricity, co-ordinates of the foci S and S', and the equations of the directrices. **3**

(ii) Find the equation of the tangent to the ellipse at a point $P(3\cos\theta, 2\sin\theta)$ on it, where θ is the auxiliary angle. **2**

(iii) The ellipse meets the *y* axis at the points *A* and *B*. The tangents to the ellipse at *A* and *B* meet the tangent at *P* at the points *C* and *D* respectively. **4**

Prove that $AC.BD=9$.

(b) (i) If ω is the root of $z^5-1=0$ with the smallest positive argument, find the real quadratic equation with roots $\omega+\omega^4$ and $\omega^2+\omega^3$. **4**

(ii) Given that $z=X+iY$ and $w=x+iy$ where $z=w^n$ for positive integers *n*, prove that $X^2+Y^2=\left(x^2+y^2\right)^n$. **2**

Question 6. (15 marks) (Start a new answer sheet.)

(a) Prove that the curve $\sqrt{\frac{x}{u}}+\sqrt{\frac{y}{v}}=1$ touches the y axis (u and v are positive constants). **2**

(b) A body of unit mass is projected vertically upwards against a constant gravitational force g and a resistance $\frac{v}{10}$, where v is the velocity of the projectile at a given time t. The initial velocity is $10(20-g)$.

(i) Show that the equation of motion of the projectile is $\frac{dv}{dt}=-\frac{v}{10}-g$. **1**

(ii) Prove that the time T for the particle to reach its greatest height is given by **4**

$$T=10\ln\left(\frac{20}{g}\right).$$

(iii) Show that the maximum height H is given by **4**

$$H=2000-100g\left[1+\ln\left(\frac{20}{g}\right)\right].$$

(iv) The particle falls to its original position under gravity and under the same law of resistance.

(α) What is its terminal velocity? **2**

(β) Will the time taken to reach the maximum height be greater or less than the time taken to fall to the original position from the maximum height? (Give reasons for your answer.) **2**

Question 7. (15 marks) (Start a new answer sheet.)

(a) Assume that ten distinct points are drawn on a number plane, no three of which are collinear. Find:

(i) The maximum number of lines that can be drawn through these points if there are no restrictions other than that each line must include two points. **1**

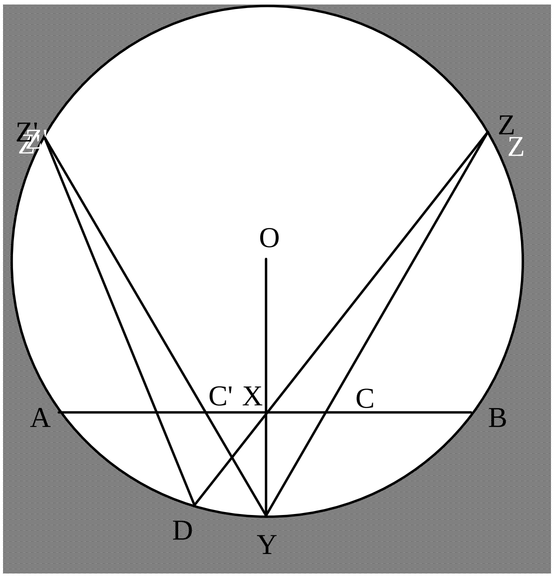
(ii) How many diagonals a convex decagon would have if these ten points were the vertices of the decagon? **2**

(b) If the roots of the polynomial equation $x^n - 1 = 0$ are $1, \alpha_1, \alpha_2, \ldots \alpha_{n-1}$ prove that **3**

$$(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots(1-\alpha_{n-1}) = n\,.$$

(c) Show that if $x > 0$, then $\displaystyle\int_0^x \frac{t^{n-1}}{1+t}\,dt < \frac{x^n}{n}$, (give reasons). **3**

(d)



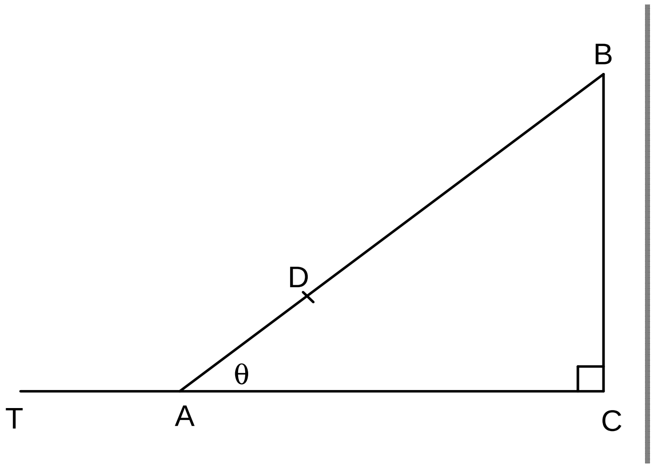
Points *A, D, Y, B, Z, Z'* lie on the circumference of the circle with centre *O*. The line *OXY* from the centre of the circle is perpendicular to the chord *AB* and meets this chord at *X*. Given also that $AC' = BC$ and $ZZ' \parallel AB$:

(i) Prove that *C', X, Y, D* are concyclic. **3**

(ii) Prove that $CY \geq XD$. **3**

Question 8. (15 marks) (Start a new answer sheet.)

(a)



The diagram represents a hill AB of uniform slope making an angle of θ with the horizontal ground. From a point B at the top of the hill the angle of depression of a point T on the ground below is 30^{o} and from a point D, three-quarters of the way down the slope the angle of depression of the point T is 15^{o}. **5**

Show that the slope of the hill is given by: $\cot\theta = \sqrt{3} - \frac{2}{3}$.

(b) (i) State the binomial theorem for $(1+x)^n$ where n is a positive integer. **2**

(ii) Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$ **2**

(iii) Show by using mathematical induction or otherwise that $\frac{1}{n!} < \frac{1}{2^{n-1}}$ for integers n where $n \geq 3$. **3**

(iv) Deduce that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = N$ where $2 < N < 3$. **3**

***This is the end of the paper*.**

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

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1998

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CERTIFICATE EXAMINATION

Mathematics Extension 2

Sample Solutions

(a) (i) $\int \frac{dx}{\sqrt{4-9x^2}}$

$= \int \frac{dx}{3\sqrt{\frac{4}{9}-x^2}}$ (2)

$= \frac{1}{3}\sin^{-1}\frac{3x}{2} + c$

(ii) $\int \frac{1}{x}(1+\ln x)^5\,dx$

$u = 1+\ln x$ (3)

$du = \frac{1}{x}\,dx$

$\int u^5\,du = \frac{1}{6}u^6 + c$

$= \frac{(1+\ln x)^6}{6} + c.$

(b) $\int_0^{\pi/4} x\sec^2 x\,dx$

$= \int_0^{\pi/4} x\frac{d}{dx}(\tan x)\,dx.$

$= \left[x\tan x\right]_0^{\pi/4} - \int_0^{\pi/4}\tan x\,dx$

$= \pi/4 + \int_0^{\pi/4}\frac{(-\sin x)}{\cos x}\,dx$

$= \pi/4 + \left[\ln|\cos x|\right]_0^{\pi/4}$

$= \pi/4 + \ln 1/\sqrt{2}$

$= \pi/4 - \frac{1}{2}\ln 2$ (3)

(c) $\int_0^5 \frac{x}{\sqrt{x+4}}\,dx$

let $u = x+4$, $x = u-4$

$du = dx$

$x=0, u=4.\quad x=5, u=9.$

$= \int_4^9 \frac{(u-4)\,du}{u^{1/2}}$

$= \int_4^9 \left(u^{\frac{1}{2}} - 4u^{-1/2}\right)du$

$= \left[\frac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}}\right]_4^9$

$= (18-24) - \left(\frac{16}{3} - 16\right)$

$= 14/3 = 4\,2/3.$ (3)

(d) let

$\frac{8}{(x+2)(x^2+4)} = \frac{a}{x+2} + \frac{bx+c}{x^2+4}.$

$8 = (a+b)x^2 + (2b+c)x + (4a+2c)$

equate coeff. of x^2, x

$a+b=0 \Rightarrow b=-a$ — (1)

$2b+c=0 \Rightarrow -2a+c=0$

i.e. $c = 2a$ — (2)

$4a+2c = 8$

$4a+4a=8 \Rightarrow a=1$

$\therefore b=-1,\ c=2.$

$\therefore \int_0^2 \frac{8}{(x+2)(x^2+4)}\,dx$ (4)

$= \int_0^2\left(\frac{1}{x+2} + \frac{2-x}{x^2+4}\right)dx$

$= \left[\ln|x+2| + \tan^{-1}\frac{x}{2} - \frac{1}{2}\ln(x^2+4)\right]_0^2$

$= \left(\ln 4 + \frac{\pi}{4} - \frac{1}{2}\ln 8\right)$

$- \left(\ln 2 - \frac{1}{2}\ln 4\right)$

$= 2\ln 2 + \frac{\pi}{4} - \frac{3}{2}\ln 2 - \cancel{\ln 2} + \cancel{\ln 4}$

$= \frac{1}{2}\ln 2 + \frac{\pi}{4}$

Question 2:

(a) (i) $z = 2+2i$

$|z| = \sqrt{4+4}$

$= 2\sqrt{2}$

$\arg z = \frac{\pi}{4}$.

$\therefore z = 2\sqrt{2}\ \text{cis}\ \frac{\pi}{4}$. (2)

(ii) $z^8 = \left(2\sqrt{2}\ \text{cis}\ \frac{\pi}{4}\right)^8$

$= (2\sqrt{2})^8\ \text{cis}\ 2\pi$

$= 256 \times 16$

$= 4096$

$= 4096 + 0.i$ (2)

b) z is $(1+i\sqrt{2})\ \text{cis}\ \frac{\pi}{3}$

$= (1+i\sqrt{2})\left(\frac{1}{2} + i.\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2} - \frac{\sqrt{6}}{2} + i\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2}\left(1-\sqrt{6} + i(\sqrt{2}+\sqrt{3})\right)$ (2)

$Re(z) = |z|$

$\therefore x = \sqrt{x^2+y^2} \Rightarrow x > 0$

$\therefore x^2 = x^2+y^2$

$\therefore y^2 = 0$

$\therefore y = 0$

which is the +ve Real axis

i.e. z is a +ve real number. (2)

$Im\left(\frac{a+ib}{c+id}\right)$

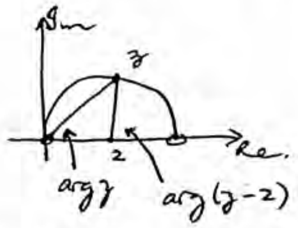
$= Im\left(\frac{(a+ib)(c-id)}{c^2+d^2}\right)$

$= Im\left(\frac{ac+bd+i(bc-ad)}{c^2+d^2}\right)$

$= \frac{bc-ad}{c^2+d^2}$ (2)

< 0 as $ad > bc$.

(e) $|z-2| = 2$ and $0 < \arg z < \frac{\pi}{2}$.



(i) $|z^2-2z| = |z||z-2|$

$= 2|z|$ (2)

(ii) $\arg(z-2) = k\arg(z^2-2z)$

$\therefore \arg(z-2) = k(\arg z + \arg(z-2))$

$\therefore \arg(z-2)(1-k) = k\arg z$.

$\therefore 2\arg z(1-k) = k\arg z$

(angle at centre = 2 x angle at circumference)

$\therefore 2(1-k) = k$.

$\therefore 2-2k = k$

$\therefore 3k = 2$

$\therefore k = \frac{2}{3}$. (3)

15

3. (a) $2x^3 = -5x - 1$

$$2\alpha^3 = -5\alpha - 1$$
$$2\beta^3 = -5\beta - 1$$
$$2\gamma^3 = -5\gamma - 1$$
$$2(\alpha^3+\beta^3+\gamma^3) = -5(\alpha+\beta+\gamma) - 3$$
$$2(\alpha^3+\beta^3+\gamma^3) = -5\times 0 \quad -3$$
$$\alpha^3+\beta^3+\gamma^3 = -\frac{3}{2}$$

2

(b) $P(x) = 2x^3 - 4x^2 + mx + n$

Let roots be $1+i$, $1-i$, α

Σ roots $\quad 2+\alpha = 2$

$\alpha = 0 \qquad n = 0$

4

Σ two at a time $= 2 + 2\alpha = \frac{m}{2}$

$m = 4$

(c) $P(x) = ax^3 + bx^2 + cx + d = (x^2-9)(x+\alpha) + x + 8$

$P(0) = -4 \qquad \underline{d = -4} \qquad a = 1$

subst

$x = 3 \qquad 27 + 9b + 3c - 4 = 11$

$x = -3 \qquad -27 + 9b - 3c - 4 = 5$

$54 + 6c = 6$

$6c = -48$

$\underline{c = -8}$

3

$27 + 9b - 24 - 4 = 11$

$9b - 1 = 11$

(d) (i) $\cos 4\theta = \cos 3\theta$.

$$\begin{aligned}\cos 4\theta &= \cos(2(2\theta)) \\ &= 2\cos^2 2\theta - 1 \\ &= 2\left(2\cos^2\theta - 1\right)^2 - 1 \\ &= \cancel{4\cos^2\theta - 2 - 1} \\ &= 2\left(4\cos^4\theta - 4\cos^2\theta + 1\right) - 1 \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta) \\ &= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

2

$$8\cos^4\theta - 8\cos^2\theta + 1 = 4\cos^3\theta - 3\cos\theta$$
$$8\cos^4\theta - 4\cos^3\theta - 8\cos^2\theta + 3\cos\theta + 1 = 0$$

(ii) $\cos 4\theta = \cos 3\theta$

$4\theta = \pm 3\theta + 2n\pi$ (General soln).

2

$\therefore 7\theta = 2n\pi$ or $\theta = 2n\pi$

$\therefore \theta = \frac{2n\pi}{7}$

$\therefore \theta = \frac{2n\pi}{7}$ satisfies the equation

(iii) $8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0$ 2

$(x+1)\left(8x^3 + 4x^2 - 4x - 1\right) = 0$

Required polynomial.

$$2+x \overline{\big) \; 2-x \;}\quad \text{quotient } -1, \quad \frac{-(-2-x)}{4}$$

(a)

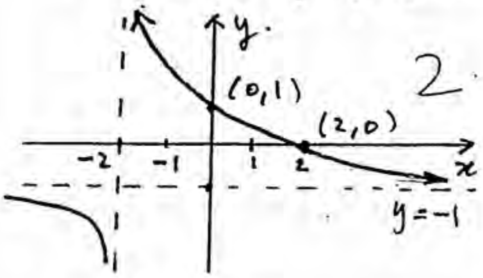
(i) $\therefore \quad \dfrac{2-x}{2+x} = -1 + \dfrac{4}{2+x}$.

When $x = 0,\ y = 1$

$x = 2,\ y = 0$.

$$\lim_{x \to \pm\infty} \left(\frac{2-x}{2+x}\right) = -1.$$

$\therefore$ graph of $y = f(x)$:



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(ii) $y = \left(\dfrac{2-x}{2+x}\right)^2$.

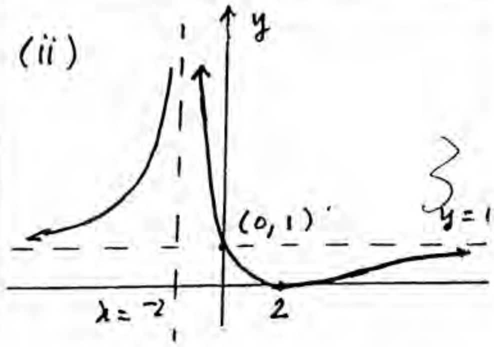
$$\frac{dy}{dx} = 2\left(\frac{2-x}{2+x}\right) \cdot \left[\frac{-(2+x)-(2-x)}{(2+x)^2}\right]$$

$$= \frac{-8(2-x)}{(2+x)^3}.$$

x	1	2	3
$\frac{dy}{dx}$	−	0	+

Note: The x-intercepts and the x-coordinates of stationary points of $y = f(x)$ give the stationary points of $y = [f(x)]^2$.

(ii)



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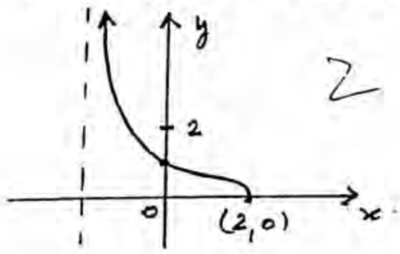
(iii) x intercept of $y = \dfrac{2-x}{2+x}$ is $(2,0)$

$\therefore (2,0)$ is a critical point of $y = \sqrt{\dfrac{2-x}{2+x}}$.



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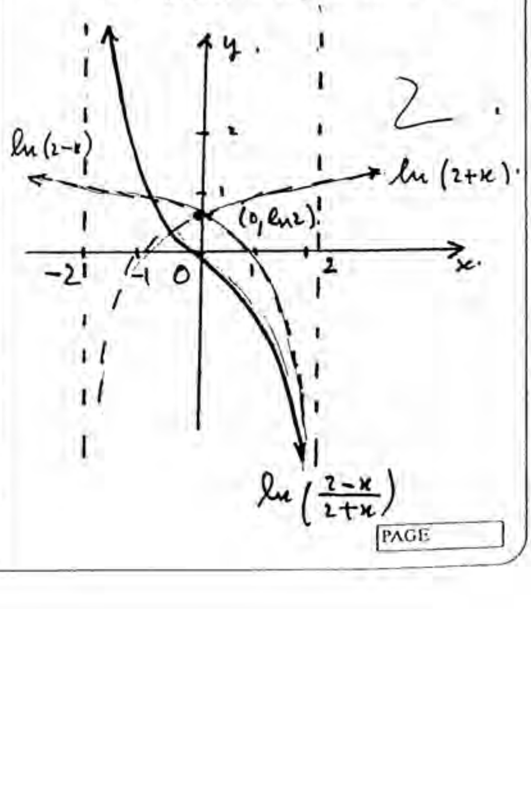
(iv) $y = \ln\left(\dfrac{2-x}{2+x}\right)$

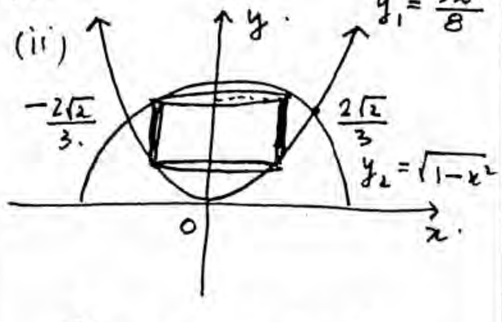
$= \ln(2-x) - \ln(2+x)$.



2

(b)

(ii)



$$\delta V = 2\pi x y \cdot \delta x$$

$$= 2\pi x (y_2 - y_1)\,\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{2\sqrt{2}/3} 2\pi x\left(\sqrt{1-x^2} - \frac{3x^2}{8}\right)\delta x$$

$$= 2\pi \int_0^{2\sqrt{2}/3} \left(x\sqrt{1-x^2} - \frac{3x^3}{8}\right) dx$$

$$= -\pi \int_0^{\frac{2\sqrt{2}}{3}} (-2x)\sqrt{1-x^2}\, dx - \frac{3\pi}{4}\int_0^{2\sqrt{2}/3} x^3\, dx$$

4

Now, $-\pi \int_0^{\frac{2\sqrt{2}}{3}} (-2x)\sqrt{1-x^2}\, dx$.

$$\boxed{\begin{array}{l} \text{let } u = 1 - x^2 \\ \quad du = -2x\,dx \\ x = 0, \; u = 1 \\ x = \frac{2\sqrt{2}}{3}, \; u = \frac{1}{9} \end{array}}$$

$$= -\pi \int_1^{1/9} u^{\frac{1}{2}}\, du$$

$$= \pi \int_{1/9}^{1} u^{\frac{1}{2}}\, du$$

$$= \left[\frac{2\pi}{3} u^{\frac{3}{2}}\right]_{1/9}^{1}$$

$$= \frac{52\pi}{81}$$

$$\frac{3\pi}{4}\int_0^{\frac{2\sqrt{2}}{3}} x^3\, dx$$

$$= \left[\frac{3\pi}{16} x^4\right]_0^{\frac{2\sqrt{2}}{3}}$$

$$= \frac{\cancel{3}\pi}{\cancel{16}} \times \frac{\cancel{64}^{\,4}}{\cancel{81}_{\,27}}$$

$$= \frac{4\pi}{27}$$

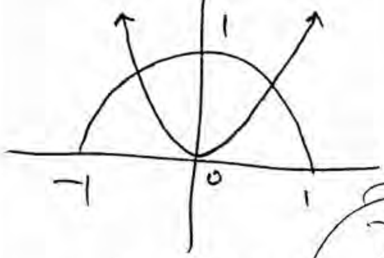
$$= \frac{12\pi}{81}$$

$$\therefore V = \frac{(52 - 12)\pi}{81}$$

$$= \frac{40\pi}{81}$$

Q(4)

(b) (i)



2

$$x^2 = \frac{8}{3}y$$

$$3y^2 + 8y - 3 = 0$$

$$\therefore y = \frac{-8 \pm \sqrt{100}}{6}$$

$$\therefore y = \frac{-8 \pm 10}{6} = \frac{1}{3} \text{ or } \cancel{-3}$$

$$x^2 = 8/9, \quad x = \pm\frac{2\sqrt{2}}{3}$$

Question 5:

(a) $\frac{x^2}{9}+\frac{y^2}{4}=1$

$\therefore \frac{b^2}{a^2}=1-e^2$

$\therefore e^2 = 1-\frac{4}{9}$

$\therefore e^2 = \frac{5}{9}$

(i) $\therefore e = \frac{\sqrt5}{3}$ (e = eccentricity)

Foci are $(\sqrt5, 0)$ and $(-\sqrt5, 0)$

Directrices are $x = \pm \frac{3}{\left(\frac{\sqrt5}{3}\right)}$

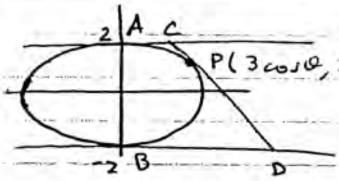
$\therefore x = \pm \frac{9}{\sqrt5}$ (3)

(ii) Equation of tangent is

$\frac{x \cdot 3\cos\theta}{9}+\frac{y \cdot 2\sin\theta}{4}=1.$

$\therefore \frac{x\cos\theta}{3}+\frac{y\sin\theta}{2}=1.$ (2)

(iii)



If $y=2$: $\frac{x\cos\theta}{3}+\sin\theta=1$

$\therefore \frac{x\cos\theta}{3}=1-\sin\theta$

$\therefore x=\frac{3(1-\sin\theta)}{\cos\theta}$

$\therefore AC = \frac{3(1-\sin\theta)}{\cos\theta}$

If $y=-2$: $\frac{x\cos\theta}{3}-\sin\theta=1.$

$\therefore x = \frac{3(1+\sin\theta)}{\cos\theta}$

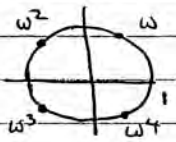
$\therefore BD = \frac{3(1+\sin\theta)}{\cos\theta}.$

$\therefore AC.BD = \frac{3(1-\sin\theta)}{\cos\theta}\cdot\frac{3(1+\sin\theta)}{\cos\theta}$

$= \frac{9(1-\sin^2\theta)}{\cos^2\theta}.$ (4)

$= 9.$

(b) $z^5-1=0$



$z^2+Az+B=0$

where $(\omega+\omega^4)+(\omega^2+\omega^3)=-A.$

$(\omega+\omega^4)(\omega^2+\omega^3)=B.$

from $z^5-1=0$: $1+\omega+\omega^2+\omega^3+\omega^4=0$

$\therefore \omega+\omega^2+\omega^3+\omega^4=-1$

$\therefore A=1.$

$(\omega+\omega^4)(\omega^2+\omega^3)=\omega^3+\omega^4+\omega^6+\omega^7$

$=\omega^3+\omega^4+\omega+\omega^2$

$=-1.$

$\therefore B=-1.$ (4)

$\therefore$ Eqn is $z^2+z-1=0.$

(ii) $z=X+iY$ $\quad \omega = x+iy.$

$z=\omega^n.$

$\therefore |z|=|\omega^n|$

$\therefore |z|^2=(|\omega|^n)^2$

$\therefore X^2+Y^2=(|\omega|^2)^n$ (2)

$=(x^2+y^2)^n.$

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$$\sqrt{\frac{x}{u}} + \sqrt{\frac{y}{v}} = 1$$

When $x = 0$ $y = v$

Diff implicitly w r t x

$$\frac{1}{2}\left(\frac{x}{u}\right)^{-\frac{1}{2}} + \frac{1}{2}\left(\frac{y}{v}\right)^{-\frac{1}{2}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\left(\frac{x}{u}\right)^{-\frac{1}{2}}}{\left(\frac{y}{v}\right)^{-\frac{1}{2}}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{\frac{y}{v}}}{\sqrt{\frac{x}{u}}} = -K\sqrt{\frac{y}{x}}$$

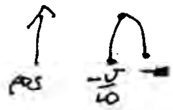
As $x \to 0$, $\frac{dy}{dx} \to -\infty$

$\therefore$ Tangent is vertical at $x = 0$

$\therefore$ The curve touches the Y axis at $(0, v)$.

Question 6

(i) $$\ddot{x} = -\frac{v}{10} - g$$



$$\therefore \frac{dv}{dt} = -\frac{v+10g}{10}$$

(ii) $$\frac{dt}{dv} = -\frac{10}{v+10g}$$

$$t = -10\ln(v+10g) + c$$

When $t=0$, $v = 200-10g$: $\quad 0 = -10\ln(200-10g+10g) + c$

$$c = 10\ln 200$$

$$t = 10\ln\frac{200}{v+10g}$$

When $v=0$: $\quad T = 10\ln\frac{200}{10g} = 10\ln\left(\frac{20}{g}\right)$

(iii) $$v\frac{dv}{dx} = -\frac{10g+v}{10}$$

$$\frac{dx}{dv} = -\frac{10v}{10g+v}$$

$$\frac{dx}{dv} = -10$$

$$\frac{dx}{dv} = -10\,\frac{v+10g-10g}{v+10g}$$

$$\frac{dx}{dv} = -10 + \frac{100g}{v+10g}$$

$$x = -10v + 100g\ln(v+10g) + c$$

When $x=0$, $v = 200-10g$: $\quad 0 = -2000 + 100g + 100g\ln 200 + c$

$$x = -10v + 100g\ln(v+10g) + 2000 - 100g - 100g\ln 200$$

When $v=0$

$$x = 2000 - 100g\left[1 + \ln\frac{200}{10g}\right]$$

$$x = 2000 - 100g\left[1 + \ln\left(\frac{20}{g}\right)\right]$$

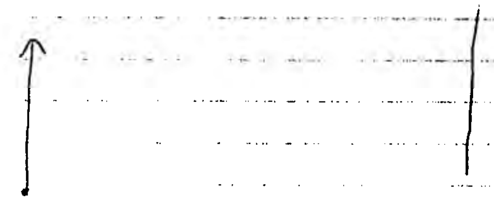
(iv) (α)

$$\ddot{x} = 0$$

$$0 = -\frac{v}{10} + g$$

$$v = 10g.$$

(β)



$v = 200 - 10g$

$> 10g$

Magnitude of acceleration is greater for upward path than for downward path.
Initial velocity up is greater than terminal velocity down.
Hence time for upward journey must be shorter.

Q (7)

(a) (i) ${}^{10}C_2$

(ii) ${}^{10}C_2 - 10$

(b) $x^n - 1 = 0$

i.e

$(x-1)(x^{n-1} + x^{n-2} + \cdots + 1) = 0.$

$(x^{n-1} + x^{n-2} + \cdots + 1)$

$= (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1})$

let $x = 1$

$\therefore n = (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n-1}).$

(c) $\int_0^n \frac{t^{n-1}}{1+t}\,dt < \frac{x^n}{n}.$

Now $x > 0 \Rightarrow t > 0$

$\int_0^x \frac{t^{n-1}}{1+t} < \int_0^x t^{n-1}\,dt$

$\because \frac{t^{n-1}}{1+t} < t^{n-1} \ \forall t$ except $t = 1$

$\therefore \int_0^x \frac{t^{n-1}}{1+t}\,dt < \int_0^x t^{n-1}\,dt$

Where $0 \le t \le x$.

(d)

$D\hat{Y}C' = D\hat{Y}Z'$

(identical angles.

$= \angle Z'ZD$

($\angle$ subtended at circumference by DZ')

$= \angle ZXC$

(alt. angles $ZZ' \parallel AB$)

$= \angle C'XD$ (vert. opp.

$\therefore$ concyclic.

(ii) $C'\hat{X}Y$ is a rt. angle,

$\Rightarrow C'Y$ is a diameter of circle $C'DYX$,

& except when $\angle DC'X$ is also a right angle, DX is only a chord of the same circle.

$\therefore C'Y \ge XD$

$\Rightarrow CY \ge XD$

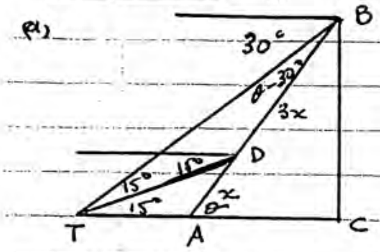
Since $C'Y = CX$

$\triangle C'XY \equiv \triangle CXY$

(SAS.

Question 8:

(a)



$\frac{x}{\sin 15°} = \frac{AT}{\sin(\theta-15°)} \Rightarrow AT = \frac{x\sin(\theta-15°)}{\sin 15°}$

$\frac{4x}{\sin 30°} = \frac{AT}{\sin(\theta-30°)} \Rightarrow AT = \frac{4x\sin(\theta-30°)}{\sin 30°}$

$\therefore \frac{\sin(\theta-15°)}{\sin 15°} = \frac{4\sin(\theta-30°)}{\sin 30°}$

$\therefore \frac{\sin\theta\cos 15° - \cos\theta\sin 15°}{\sin 15°} = 8\left(\frac{\sin\theta\cos 30° - \cos\theta}{\sin 30°}\right)$

$\therefore \sin\theta\cot 15° - \cos\theta = 4(\sqrt{3}\sin\theta - \cos\theta)$

Now, $\tan 30° = \frac{2\tan 15°}{1-\tan^2 15°}$

$\therefore \frac{1}{\sqrt{3}} = \frac{2\tan 15°}{1-\tan^2 15°}$

$\therefore \tan^2 15° + 2\sqrt{3}\tan 15° - 1 = 0$

$\therefore \tan 15° = \frac{-2\sqrt{3} \pm \sqrt{12-4\times1\times-1}}{2}$

$= 2-\sqrt{3}$ (as $\tan 15° > 0$)

$\therefore \cot 15° = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}$

$\therefore \sin\theta(2+\sqrt{3}) - \cos\theta = 4\sqrt{3}\sin\theta - 4\cos\theta$

$\therefore 3\cos\theta = \sin\theta(3\sqrt{3}-2)$

$\therefore \cot\theta = \sqrt{3} - \frac{2}{3}$

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b) (i) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$

$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + x^n$

2

(ii) $\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \dots + \frac{n(n-1)\dots1}{n!}\cdot\frac{1}{n^n}$

$= 1 + 1 + \frac{1\cdot\left(1-\frac{1}{n}\right)}{2!} + \frac{1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \dots$

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$

2

(iii) Aim: Show $\frac{1}{n!} < \frac{1}{2^{n-1}}$ for $n \geq 3$.

i.e $S(n) \equiv 2^{n-1} < n!$

Step 1: Show $S(3)$ is true

i.e $2^2 < 3!$

$LHS = 4 \quad RHS = 6$

$\therefore S(3)$ is true

Step 2: Assume $S(k)$ is true

i.e $2^{k-1} < k!$

Prove $S(k+1)$ is true

i.e $2^k < (k+1)!$

$LHS = 2.2^{k-1} < 2.k!$

$< (k+1).k!$

$= (k+1)!$

$\therefore$ If $S(k)$ is true, $S(k+1)$ is true

Step 3: $S(3)$ is true $\therefore$ By Step 2 $S(4)$ is true

$S(4)$ is true $\therefore$ By Step 2 $S(5)$ is true

and so on for all integral $n \geq 3$.

3

(iv) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$

$= 2\frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \dots$

$< 2\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

$= 2\frac{1}{2} + \frac{\frac{1}{4}}{1-\frac{1}{2}}$

$= 3$

$\therefore N > 2\frac{1}{2}$ and $N < 3$

3

Hence $2 < N < 3$.